

Characters of Feigin-Stoyanovsky subspaces and Brion's theorem

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Abstract

We give an alternative proof of the main result of paper [1], the proof relies on Brion's theorem about convex polyhedra. The result itself can be viewed as a formula for the character of the Feigin-Stoyanovsky subspace of an integrable irreducible representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_n(\mathbb{C})$. Our approach is to assign integer points of a certain polytope to the vectors comprising a monomial basis of the subspace and then compute the character via (a variation of) Brion's theorem.

1. Introduction

Within this section it will be convenient to set $a_n = a_0$.

We begin by introducing the necessary notation and then giving the statement of the result in [1].

First consider the simple Lie algebra $L = \mathfrak{sl}_n(\mathbb{C})$ with a fixed Cartan decomposition $L = N^- \oplus H \oplus N^+$, as well as a set of simple roots $\alpha_1, \dots, \alpha_{n-1} \in H^*$ with the standard numbering. For a positive root α consider the generators e_α, f_α in the root spaces of α and $-\alpha$.

We will be interested in the representations of the corresponding affine Lie algebra

$$\hat{L} = \widehat{\mathfrak{sl}}_n(\mathbb{C}) = L \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is the central element and d is the degree operator. The theory of such algebras and their representations is presented, for example, in [4].

The following notation will be useful to us: let $x(m)$ denote $x \otimes t^m \in \hat{L}$ for $x \in L$.

We will be working with the integrable irreducible representation $L(\lambda)$ of \hat{L} with highest weight λ and highest weight vector v_0 . The weight λ can be written as

$$(a_0, \dots, a_{n-1})$$

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with respect to the basis consisting of the fundamental weights. In other words, $\lambda(h_{\alpha_i}(0)) = a_i$ for each $1 \leq i \leq n-1$ and

$$\lambda(c) = k = \sum_{i=0}^{n-1} a_i$$

(k is the level of $L(\lambda)$). Weight λ is integral and dominant, thus all the a_i are nonnegative integers.

Now we define the subspace $V \subset L(\lambda)$, sometimes referred to as the Feigin-Stoyanovsky subspace. Let

$$\{\gamma_i = \alpha_1 + \dots + \alpha_i\}$$

be a basis of the root space of L and denote $f_i = f_{\gamma_i}$. Let P be the abelian subalgebra of \hat{L} , generated by all the $f_i(m)$ with $m \leq 0$. The subspace of interest is $V = \mathcal{U}(P)(v_0)$.

A monomial basis in V is constructed in [1], which can be defined as follows. The monomials in $\mathcal{U}(P)$ correspond to sequences of nonnegative integers with finite support. Namely, for a monomial p and any integers $q \geq 0$ and $1 \leq r \leq n-1$ the term $x(p)_{q(n-1)+r}$ of the corresponding sequence $(x(p)_i)_{i \geq 1}$ is equal to the power of $f_r(-q)$ in p . Simply put, the terms of the sequence are just the powers of the monomial with the $f_i(-m)$ ordered lexicographically by m and i .

Now we introduce a sequence ψ_i of functionals on the space of sequences with finite support. For $i < n$ let $\psi_i(x) = x_1 + \dots + x_i$ while $\psi_i(x) = x_{i-n+1} + \dots + x_i$ for $i \geq n$. The mentioned basis is given by the set of monomials $\Pi \subset \mathcal{U}(P)$, for which the sequence $x(p)$ satisfies the following set of inequalities.

A. $\psi_i(x(p)) \leq a_1 + \dots + a_i =: b_i$ for each $1 \leq i \leq n-1$.

B. $\psi_i(x(p)) \leq k =: b_i$ for each $i \geq n$.

Thus we can state

Theorem 1.1. The set $\{pv_0, p \in \Pi\}$ comprises a basis of the space V .

If $S \subset \mathcal{U}(P)$ is the span of the monomials $p \in \Pi$, then $p \rightarrow pv_0$ provides a linear map $\varphi : S \rightarrow V$. Theorem 1.1 is proved in [1] by showing that φ is surjective and $\text{char } V = \exp \lambda \text{char } S$. In order to prove the last equality, an explicit formula for the character $\text{char } S$ is given. That computation is the focal point of [1]. In this paper we essentially present a different method of obtaining the formula. We now proceed to give the necessary definitions and formulate the result.

Denote by Θ the set of *good* binary sequences $y = (y_i)$ (of 0's and 1's) with finite support. A sequence is considered good if for every i one has $y_{i+n-1} = 0$ whenever $y_i = 0$. Let us define a character F_y for each good y , the result will be given in terms of these F_y .

Each good sequence is associated with an element of W , the Weyl group of \hat{L} . Let $s_\alpha \in W$ be the reflection corresponding to root α . Furthermore, for integers $1 \leq r \leq n-1$ $q \geq 0$ let $\gamma_{q(n-1)+r} = \gamma_r - q\delta$. Then we simply set

$$w_y = \dots s_{\gamma_i}^{y_i} \dots s_{\gamma_1}^{y_1}.$$

There is also an $\xi \in W$ mapping each γ_i to γ_{i+1} (defined in [1]). The characters F_y satisfy a recurrent definition.

$$\begin{aligned} F_{(0,0,0,\dots)} &= \frac{1}{\prod_{i \geq 1} (1 - \exp \gamma_i)}, \\ F_{(y_1, y_2, \dots)} &= \frac{\xi F_{(y_2, y_3, \dots)}}{1 - \exp(w_{(y_1, y_2, \dots)} \gamma_1)}, \end{aligned} \tag{1}$$

with the Weyl group acting on characters in the natural way.

We are finally ready for the character formula.

Theorem 1.2.

$$\text{char } S = \exp(-\lambda) \sum_{y \in \Theta} \exp(w_y \lambda) F_y.$$

The approach in [1] is to show that both the left- and the right-hand side are the sole solution of the same recurrence equation. Ours, on the other hand, makes use of the following observation. The set of sequences $x(p), p \in \Pi$ can be viewed as the set of integral points of a convex polytope in countable dimensional space. This polytope will also be denoted as Π . The character $\text{char } S$ is, in turn, the sum of certain exponents of these integral points. Such sums for finite dimensional polytopes can be computed via Brion's theorem. It'll be shown that a similar identity holds for Π and thus gives a formula for $\text{char } S$.

2. Brion's Theorem

Consider the space \mathbb{R}^m with a fixed basis determining the subset \mathbb{Z}^m of integer points. To each such point we associate its *exponent*, the Laurent monomial $\exp(x) = t_1^{x_1} \dots t_m^{x_m}$ in formal variables t_1, \dots, t_m . For a subset $\Sigma \subset \mathbb{R}^m$ we define its *generating function* as the Laurent series

$$S(\Sigma) = \sum_{x \in \Sigma \cap \mathbb{Z}^m} \exp(x).$$

Now let C be a *rational polyhedral cone*, a cone with rational vertex and finite set of rational generators. The series $S(C)$ can be viewed as a rational function in the following sense. Considering the $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ -module structure on the space of Laurent series, it can be seen that there exists a polynomial θ such that $\theta S(C)$ is also a polynomial (i.e. has a finite number of nonzero coefficients). Moreover, the rational function $\frac{\theta S(C)}{\theta}$ does not depend on the choice of θ and will be referred to as $\sigma(C)$.

A more detailed description of $\sigma(C)$ will be useful to us. First, let C be a simplicial cone with vertex v and linearly independent set of integral generators u_1, \dots, u_l . Each of these generators is chosen to be minimal, i.e. have setwise coprime coordinates. The set Σ of points

$$v + \sum_{i=1}^l \alpha_i u_i, \text{ all } \alpha_i \in [0, 1)$$

is the *fundamental parallelepiped* of C . In these terms one has

$$\sigma(C) = \frac{\sum_{x \in \Sigma \cap \mathbb{Z}^m} \exp x}{(1 - \exp u_1) \dots (1 - \exp u_l)}. \quad (2)$$

A non-simplicial cone C can be triangulated into simplicial ones. The function $\sigma(C)$ can then be expressed by the arising inclusion-exclusion formula. An important case is the case of a *degenerate* cone, a cone containing an affine line (any affine space or semispace, for instance). For a degenerate rational cone one has $\sigma(C) = 0$.

A detailed discussion of these subjects can be found in the book [5].

Finally, let Σ be a rational polyhedron, the convex hull of a finite set of rational points. Its vertex cones C_i are the cones with vertices in the vertices of Σ and generators given by the edges of Σ adjacent to the corresponding vertex. The observation that in this case the series $S(\Sigma)$ is actually a polynomial and thus a rational function lets us state

Theorem 2.1 (M. Brion, [2]). In the above notation the identity

$$S(\Sigma) = \sum_i \sigma(C_i)$$

holds in the field of rational functions.

Let us now prove another identity, closely related to Brion's theorem. We generalize the notion of vertex cones by defining an associated cone C_Γ for face Γ of arbitrary dimension of polytope Σ in \mathbb{R}^m . (We use the term "polytope" to refer to the intersection of any finite set of semispaces, a polyhedron is a bounded polytope.) Let x be a point in the interior of Γ , then the cone is defined as

$$C_\Gamma = \{x + \alpha(y - x), y \in \Sigma, \alpha \geq 0\}.$$

Note that C_Γ is degenerate whenever $\dim \Gamma > 0$.

Theorem 2.2. Consider a rational cone C as well as the cones C_Γ defined above for all faces of C . Also, let D be a rational cone with the same vertex as C , let D contain C and, furthermore, suppose that D is not all of \mathbb{R}^m . Then

$$\sigma(C) = \sum_{\dim \Gamma > 0} (-1)^{\dim \Gamma + 1} \sigma(C_\Gamma \cap D).$$

Proof. If C is degenerate then so is D and all of the $C_\Gamma \cap D$ and both sides are equal to zero. We now suppose C to be non-degenerate.

By $\mathbb{1}_\Sigma$ we denote the characteristic function of $\Sigma \subset \mathbb{R}^m$. Let \hat{C} be the interior of the image of C under symmetry across its vertex. One sees that \hat{C} is precisely the set of points not contained in any C_Γ other than C_C . The main ingredient of our proof is the following lemma.

Lemma 2.1. The functions $\mathbb{1}_C$ and

$$\Phi = \sum_{\dim \Gamma > 0} (-1)^{\dim \Gamma + 1} \mathbb{1}_{C_\Gamma}$$

coincide on the complement of \hat{C} .

Proof of the lemma. Obviously both functions are zero outside of the affine hull of C . Thus we may assume the affine hull to be the whole space \mathbb{R}^m .

For a point $x \notin \hat{C}$ there are two possibilities.

1. The point is not contained in the boundary of the open set \hat{C} . Visibly, since C is non-degenerate, x is contained in a hyperplane whose intersection with C is a (bounded!) polyhedron of dimension $m - 1$. The equality $\Phi(x) = \mathbb{1}_C(x)$ is then the classical Brianchon-Gram identity for the aforementioned polyhedron (cf. [3]).
2. Now, if x is contained in the boundary of \hat{C} , let y be the image of x under symmetry across the vertex. C_Γ contains x iff y is contained in Γ implying $\mathbb{1}_{C_\Gamma}(x) = \mathbb{1}_{C_\Gamma}(y)$. Thus either $y \neq x$ and we have reduced to the previous case, or $y = x$ is the vertex and the equality once again follows from Brianchon-Gram.

□

(Actually, the lemma is just as valid for degenerate cones.)
Cone D does not intersect \hat{C} and thus the lemma implies:

$$\mathbb{1}_C = \mathbb{1}_D \mathbb{1}_C = \mathbb{1}_D \sum_{\dim \Gamma > 0} (-1)^{\dim \Gamma + 1} \mathbb{1}_{C_\Gamma} = \sum_{\dim \Gamma > 0} (-1)^{\dim \Gamma + 1} \mathbb{1}_{C_\Gamma \cap D}.$$

It follows immediately that

$$S(C) = \sum_{\dim \Gamma > 0} (-1)^{\dim \Gamma + 1} S(C_\Gamma \cap D)$$

wherefrom the statement of the theorem is obvious. □

We're primarily interested in the case of D being a semispace. In this case $\dim \Gamma > 1$ implies that $C_\Gamma \cap D$ is degenerate and $\sigma(C_\Gamma \cap D) = 0$. Moreover, if u is an edge of C which is contained in the bounding hyperplane of D , then $C_u \subset D$ and again $\sigma(C_u) = 0$. We obtain

Theorem 2.3. Consider a rational polyhedral cone C with the cones C_u defined at each of its edges u . Let β be a hyperplane containing the vertex of C but none of its interior points. Let D stand for the semispace bounded by β and containing C . Then

$$\sigma(C) = \sum_{u \not\subset \beta} \sigma(C_u \cap D).$$

3. The polytope Π

Within the countable dimensional space of sequences of real numbers with finite support consider the subset of sequences whose terms are nonnegative and also satisfy conditions A and B from the introduction. This set is the polytope Π mentioned above. In this part we overview the key properties of Π and then prove an analog of Brion's theorem for the polytope.

First, let us introduce the notions of exponents, generating functions etc. in this countable dimensional setting.

Just as in the finite dimensional case, for a sequence $x = (x_i)$ one may consider its exponent, a monomial in the set of variables $\{t_i\}$. Now, apply the substitution $t_{j(n-1)+r} = z_r q^j$ for all integers $j \geq 0$ and $1 \leq r \leq n-1$, denote the result $\exp^* x$. Visibly, setting $\exp \gamma_i = z_i$ ($1 \leq i \leq n-1$) and $\exp(-\delta) = q$ in the space of characters of \hat{L} leads to the identity $\text{char } p = \exp^* x(p)$ for any monomial $p \in \mathcal{U}(P)$. This identity lets us view Laurent series in z_1, \dots, z_{n-1}, q as characters.

We arrive at $S^*(\Pi) = \text{char } S$, where $S^*(\Sigma)$ is the sum of the \exp^* exponents of integer points of Σ . (Clearly, the series $S^*(\Pi)$ is well-defined.)

Now, let C be a finite dimensional rational cone in our countable dimensional space. For some $m > 0$ the cone is contained in subspace $\{(x_i) | x_i = 0 \text{ when } i > m\}$ which lets us define the function $\sigma(C)$ in variables $\{t_i, 1 \leq i \leq m\}$. This function is, clearly, independent of the choice of m and, if the above substitution is applicable to it, we denote the result $\sigma^*(C)$.

Now we turn to the structure of Π .

We will refer to the intersections of Π with the hyperplanes given by conditions $\psi_i(x) = b_i$ (see conditions A and B) and $x_i = 0$ as its facets. We introduce the following notation: for $i > 0$ let Γ_i be the facet $\Pi \cap \{x | \psi_i(x) = b_i\}$ and Δ_i – the facet $\Pi \cap \{(x_i) | x_i = 0\}$. (At times we will take the liberty of referring to the hyperplanes themselves by the same symbols, but the intended meaning should be clear from the context.)

All of the defined sets are true facets of Π (i.e. have codimension 1) iff all of the a_i are nonzero. We will be working in the assumption that this is so throughout this and the following part, after which the case of a weight with a zero coordinate will be considered.

In analogy with finite dimensional polytopes, the vertices of Π are understood as its points for which the set of facets containing it is maximal. Here are the main properties of the vertices.

Proposition 3.1.

(a) A vertex is contained simultaneously in both Γ_i and Δ_i iff it is contained in both Γ_{i-1} and Δ_{i-n} . (In particular, $i > n$.)

(b) A point in Π is a vertex iff for each $i > 0$ it is contained in at least one of Γ_i and Δ_i .

(c) Consider a set of facets which for each i contains at least one of Γ_i and Δ_i and, furthermore, contains both of them iff it contains both Γ_{i-1} and Δ_{i-n} . If the set contains only a finite number of facets of the form Γ_i , then the intersection of all its elements is a single point which is a vertex of Π .

Proof.

(a) If $u \in \Gamma_i \cap \Delta_i$, then $\psi_{i-1}(u) = \psi(u) + u_{i-n} = b_i + u_{i-n}$. Therefore $b_{i-1} \leq b_i$ implies $u_{i-n} = 0$ (i.e. $u \in \Delta_{i-n}$) and $\psi_{i-1}(u) = b_{i-1}$ (i.e. $u \in \Gamma_{i-1}$).

(b) Let $u = (u_i)$ be a vertex, $u \notin \Gamma_l$, $u \notin \Delta_l$. From (a) one deduces that there exists a point in Π belonging to all facets containing u as well as facet Δ_l , which contradicts the maximality. The converse follows from the fact that the facets of a set containing Γ_i or Δ_i for each i have no more than one common point.

(c) Let A be such a set, $B \subset A$ contain exactly one of Γ_i and Δ_i for each i . The facets in B have exactly one common point $u = (u_i)$. Via induction on i it is easily verified that $X_i \in A \Rightarrow u \in X_i$ for all $i > 0$ and $X \in \{\Gamma, \Delta\}$. □

The edges of Π are its subsets of more than one point for which the set of facets containing it is maximal (amongst such subsets with more than one point). Each edge is a segment the endpoints of which are vertices. For an edge e and its endpoint u we define the corresponding direction vector as the minimal integral vector ε such that $u + \varepsilon \in e$.

Proposition 3.2. In the above notation the following holds.

(a) An edge is contained in both Γ_i and Δ_i iff it is contained in both Γ_{i-1} and Δ_{i-n} .

(b) The edge e is contained in at least one of the facets Γ_i and Δ_i for all i except for one.

(c) The monomial $\exp^* \varepsilon$ is of the form $\exp \alpha$ for some root of \hat{L} .

Proof.

(a) This is immediate from 3.1(b) and the fact that the endpoints of an edge are vertices.

(b) Let $l < m$ be the two minimal indices such that e is not contained in any of the facets $\Gamma_l, \Delta_l, \Gamma_m, \Delta_m$. Let u_1 be the second endpoint of e . Clearly, u_1 belongs to just one of Γ_m and Δ_m , denote that facet X . One sees that there exists a point in Π contained in all facets containing u as well as facet X . Which again contradicts the maximality property.

(c) From (a) and (b) one may deduce the following description of ε . First, all of its coordinates are either 0 or ± 1 and the nonzero coordinates are alternately 1 and -1 . Second, if i_1, \dots, i_s are the indices of nonzero coordinates, then

$$i_3 - i_2 = i_5 - i_4 = \dots = n - 1.$$

Thus, $\exp^* \varepsilon$ is $\exp(\pm(\gamma_{i_1} - \gamma_{i_s}))q^{\pm(s-2)/2}$ in case of even s and $\exp(\pm\gamma_{i_1})q^{\pm(s-1)/2}$ in case of odd s . Both expressions are exponents of a root. \square

For a vertex u we once again consider the vertex cone C_u . It is the set of points of the form

$$u + \sum_i \alpha_i \varepsilon_i,$$

where $\{\varepsilon_i\}$ is the set of direction vectors of edges containing u . Let m_u stand for the index of the last nonzero coordinate of u and l_u stand for the maximal index such that $u \in \Gamma_{l_u}$. Also, for $m \geq m_u$ let C_u^m be the section of C_u by the space $\{(x_i) | x_i = 0 \text{ when } i > m\}$, it is a finite dimensional cone with vertex u .

The following fact will be useful to us.

Proposition 3.3. All functions $\sigma(C_u^m)$ can be written as the product of $\exp^* u$ and a fraction both the numerator and denominator of which are Laurent polynomials not involving negative powers of q . Moreover, the denominator may be assumed to be the product of multiples of the form $1 - \exp \alpha$ with α a root of \hat{L} .

Proof. Consider a triangulation of C_u^m . The corresponding inclusion-exclusion formula consists of summands of the form

$$\pm \frac{\exp x}{\prod_{\varepsilon \in E} (1 - \exp \varepsilon)},$$

where E is a linearly independent set of direction vectors of edges containing u , while x is an integer point in the fundamental parallelepiped, i.e. expressible as $u + \sum_{\varepsilon \in E} \alpha_\varepsilon \varepsilon$ with $\alpha_\varepsilon \in [0, 1)$.

If $F \subset E$ is the subset of such ε that $\exp^* \varepsilon$ involves q in a negative power, then the power of q in $\exp^* x$ is no fewer than that in

$$\prod_{\varepsilon \in F} \exp^* \varepsilon. \quad (3)$$

Consequently, multiplication of the numerator and denominator by (3) provides a summand of the needed form. (We employ 3.2(c) and the fact that the edges of C_u^m are all edges of C_u as well.) \square

Within the ring $\mathbb{C}(z_1^{\pm 1}, \dots, z_{n-1}^{\pm 1}, q^{\pm 1})$ consider the subring T of functions expressible as the quotient of two Laurent polynomials, with the denominator being the product of multiples of the form $1 - \exp \alpha$ with α a root of \hat{L} . All the $\sigma^*(C_u^i)$ are elements of T . Now, let R be the ring of characters (Laurent series) the support of which is contained in the union of a finite number of lower sets of the poset of weights (cf. [4]). The ring T is embedded in R , since any $1 - \exp \alpha$ is an invertible element of R . Let τ_u^i denote the character which is the image of $\sigma^*(C_u^i)$ under this embedding.

For any $i \geq l_u$ one has

$$\sigma^*(C_u^{i+1}) = \frac{\sigma^*(C_u^i)}{1 - \exp \gamma_{i+1}},$$

hence the equality $\tau_u^{i+1}(1 - \exp \gamma_{i+1}) = \tau_u^i$ in R . Now, when $i \rightarrow \infty$ the power of variable q in $\exp \gamma_{i+1}$ approaches infinity as well, hence the coefficients of the series τ_u^i stabilize. Thus, a limit series τ_u is defined.

We are finally ready to state the sought-for analog of Brion's theorem for polytope Π .

Theorem 3.1. In R the identity

$$S^*(\Pi) = \sum_{u \text{ vertex of } \Pi} \tau_u \quad (4)$$

holds.

Proof. Let Π_m denote the section of Π by the space $\{(x_i) | x_i = 0 \text{ when } i > m\}$. The vertices of Π_m are precisely the vertices of Π with $m_u \leq m$. Thus, Brion's theorem reads as

$$S^*(\Pi_m) = \sum_{\substack{u \text{ vertex of } \Pi \\ m_u \leq m}} \sigma^*(C_u^m).$$

As an immediate consequence we obtain the equality in R :

$$S^*(\Pi_m) = \sum_{u \text{ vertex of } \Pi} \tau_u^m. \quad (5)$$

(We set $\tau_u^m = 0$ whenever $m < m_u$.)

Obviously, the coefficients of the left-hand side stabilize onto the coefficients of $S^*(\Pi)$ when m approaches infinity. Also, by definition, the coefficients of τ_u^m stabilize onto the coefficients of τ_u . However, proposition 3.3 implies that all the monomials in τ_u^m and τ_u contain q in a power of at least $\frac{m_u-1}{n-1}$. Hence, for any monomial its coefficients in both sides of (5) are the same for sufficiently large m . That is because the set of u for which the monomial's coefficient is nonzero in τ_u or some τ_m is only finite. Herefrom we obtain (4) coefficient-wise. \square

In order to compute $\text{char } S$ it will now suffice to find the characters τ_u .

4. Computation of the contributions τ_u

Let a *simple* vertex of Π be one for which the direction vectors of all the edges of the corresponding vertex cone are linearly independent (i.e. the vertex cone is simplicial). Clearly, a vertex is simple iff it is contained in exactly one of the facets Γ_i and Δ_i for each i . Theorem 3.1 enables us to express $S^*(\Pi)$ explicitly due to the following key fact.

Theorem 4.1. For a non-simple vertex u one has $\tau_u = 0$.

Proof. The following notations will be convenient. For a set X of facets of Π let X' be the set obtained from X by replacing each Γ_i with Γ_{i-1} and each Δ_i with Δ_{i-1} . (With $\Gamma_1, \Delta_1 \in X$ simply being deleted.) Further, let ξ denote the "shift" transformation substituting z_1 for z_2 , z_2 for z_3 , \dots , z_{n-1} for z_1 . This substitution may be viewed as $\xi \in W$ mentioned above acting on characters.

It will suffice to show that $\sigma^*(C_u^{l_u}) = 0$. This will be done by induction on l_u .

Apply theorem 2.3 to the cone $C_u^{l_u}$ and hyperplane β of the form $x_1 = \text{const}$ containing u . This hyperplane is, obviously, either Γ_1 or Δ_1 and contains a facet of the cone. Let $\{\varepsilon_j\}$ be the set of direction vectors of the edges of $C_u^{l_u}$ not belonging to β (i.e. having a nonzero first coordinate). Each ε_j corresponds to one of the cones C_j considered in the statement of theorem 2.3, which now reads as

$$\sigma^*(C_u^{l_u}) = \sum_j \sigma^*(C_j).$$

We will now show that the induction hypothesis implies that $\sigma^*(C_j) = 0$ whenever C_j is non-simplicial. Let $D_j = C_j \cap \beta$. There is only one edge of C_j not contained in β (the one with direction ε_j), hence it would be enough to show that $\sigma^*(D_j) = 0$ to verify the last claim.

In order to do that, consider the set X of facets containing the edge, given by ε_j . It is clear from proposition 3.2(a) together with 3.1(c) that X' is precisely the set of facets of cone $C_{u'}^{l_{u'}}$ for some vertex u' with $l_{u'} = l_u - 1$. One then observes that

$$\exp^*(-u)\sigma^*(D_j) = \xi \left(\exp^*(-u')\sigma^*(C_{u'}^{l_{u'}}) \right).$$

But the cones D_j and, consequently, $C_{u'}^{l_{u'}}$ are non-simplicial by hypothesis, hence $\sigma^*(C_{u'}^{l_{u'}}) = 0$.

We are left to show that the sum of $\sigma^*(C_j)$ over the simplicial C_j is zero. In particular, the base of our induction will follow. For such a C_j , in the above notations, the set X' contains exactly one of the facets Γ_i and Δ_i for each $1 \leq i \leq l_u - 1$. It is straightforward to deduce from propositions 3.1 and 3.2 that this is possible only if the set I of i for which $u \in \Gamma_i \cap \Delta_i$ satisfies the following description. It is of the form

$$[l_1, r_1] \cup \dots \cup [l_s, r_s]$$

with $l_1 = n + 1$ and $l_{c+1} - r_c = n$ for all $c < s$. Furthermore, the set X then comprises the facets Γ_{r_c} for all $c < s$, one (arbitrary) of Γ_{r_s} and Δ_{r_s} and Δ_i for all other $i \in I$.

Thus, in case simplicial C_j do occur, there is exactly two of them: C_{j_1}, C_{j_2} , we are to show that $\sigma^*(C_{j_1}) + \sigma^*(C_{j_2}) = 0$. This will, again, be proved by induction on l_u . Let U denote the set of all facets containing u and consider the following possibilities.

1. The value $r_1 - n + 1$ (the index of the first nonzero coordinate of u) is greater than 2. In this case we employ the induction hypothesis for vertex

u' the facets containing which are precisely elements of $U' \setminus \{\Delta_n\}$. For u' one has the two corresponding simplicial cones C'_1 and C'_2 . A calculation indicates that

$$\frac{\sigma^*(C_{j_1})}{\xi(\sigma^*(C'_1))} = \frac{\sigma^*(C_{j_2})}{\xi(\sigma^*(C'_2))},$$

which due to induction hypothesis implies the needed equality.

2. $r_1 = n+1$ and $s > 1$. We, in similar fashion, make use of the hypothesis for vertex u' the set of facets containing which is $U^{(n)} \setminus \{\Gamma_1\}$ (here $^{(n)}$ stands for the n -fold application of $'$.)
3. $r_1 = n+1$ and $s = 1$. This case is the base of our induction, the equality can be verified via direct computation.

□

We now turn to the description of simple vertices. A simple vertex u is contained in just one of Γ_i and Δ_i for each i . Hence for each i the cone C_u has exactly one such edge that the first nonzero coordinate of its direction vector ε_i has index i . The character τ_u may be expressed as

$$\tau_u = \frac{\exp^* u}{(1 - \exp^* \varepsilon_1)(1 - \exp^* \varepsilon_2) \dots}. \quad (6)$$

Proposition 3.1 shows that a simple vertex contained in Δ_i (and thus not contained in Γ_i) is necessarily contained in Δ_{i+n-1} and not in Γ_{i+n-1} . This sets up a one-to-one corresponding between the simple vertices and the good sequences of 0's and 1's. Let u_y be the vertex corresponding to good sequence y . Theorem 1.2 is then a direct consequence of

Proposition 4.1. For each good sequence y the identity $\tau_{u_y} = \exp(w_y \lambda - \lambda) F_y$ holds.

Proof. It will suffice to verify two things. First, that $\exp^* u_y = \exp(w_y \lambda - \lambda)$ and, second, that the expressions $\exp^*(-u_y) \tau_{u_y}$ satisfy the same recurrence relation (1). Formula (6) indicates that the latter would follow from $\exp^* \varepsilon_1 = \exp(w_y \gamma_1)$.

Both of these equalities are straightforward to verify via an induction on m_{u_y} , the index of the last 1 in sequence y . □

We have now completed the consideration of a weight λ for which all the a_i are nonzero. As planned, we now proceed to explain how the remaining case can be reduced to this situation.

5. Weights with a zero coordinate

Within this section it will be convenient to set $a_n = a_0$.

The main difference is that if $a_i = 0$ then a vertex of Π may be contained in both Γ_i and Δ_i even though $i \leq n$. The definitions of the sets Γ_i and Δ_i are formally the same (the intersections of Π and a hyperplane), however, they now need not have codimension 1 or be pairwise distinct. Nevertheless, the content of part 3 is easily adjusted to this case. Most importantly, we can define the τ_u in complete analogy with the above and then prove theorem 3.1.

However, the simple description of the τ_u given in part 4 is not valid anymore. There arise non-simple vertices with nonzero contributions. In order to reduce this case to the previous one we introduce the following formalism.

First consider an arbitrary weight λ_1 without zero coordinates, i.e. situated in the previous case. Proposition 3.1 shows that for any vertex of the corresponding polytope Π_1 the set of facets containing it has the following three properties.

1. For each $i > 0$ the set ω contains at least one of the facets Γ_i and Δ_i .
2. ω contains but a finite number of facets of type Γ .
3. For each $i > 0$ both Γ_i and Δ_i are in ω iff both Γ_{i-1} and Δ_{i-n} are also in ω .

(Naturally, we're referring to the facets of Π_1 .) Moreover, any set ω satisfying these conditions is precisely the set of facets containing a certain vertex of Π_1 .

Let Ω stand for the set of all sets of symbols Γ_i and Δ_i satisfying these three conditions. Ω is in bijection with the set of vertices of Π_1 . Now let u_1 be a vertex of Π_1 and τ_{u_1} be the character defined above. If $\omega \in \Omega$ is the corresponding set, then the character

$$\tau_\omega = \exp^*(-u_1)\tau_{u_1}$$

is well defined.

For our case of a singular weight λ a fourth condition is needed.

4. If $a_i = 0$, then ω contains both Γ_i and Δ_i iff it contains Γ_{i-1} or $i = 1$.

A slight adjustment of the proof of 3.1 shows that the set of sets of symbols satisfying conditions (1)-(4) is in bijection with the set of vertices of Π . This set of sets will be denoted Ω_{A_λ} where A_λ is the set of indices i for which $a_i = 0$. For $\omega \in \Omega_{A_\lambda}$ we can once again define

$$\tau_\omega = \exp^*(-u)\tau_u,$$

u being the corresponding vertex.

We can likewise define the set Ω_A for any $A \subset [1, n]$ (in particular, we have $\Omega = \Omega_\emptyset$) and the character τ_ω for any $\omega \in \Omega_A$. This is possible due to the fact that Ω_A depends only on A and not a specific weight, as well as the fact that the vertex cones of different polytopes corresponding to the same ω are identified by a translation. Given two sets $A \subset B \subset [1, n]$ for each $\omega_A \in \Omega_A$ there is a unique $\omega_B \in \Omega_B$ such that $\omega_A \subset \omega_B$. This is clear from the fact that there is unique way of supplementing some $\omega_A \in \Omega_A$ in a way for it to comply with

the new strengthened condition (4) for set B . Moreover, it is straightforward to verify that any ω_B can be acquired from some ω_A by such a supplementation. As a result we obtain a surjective map

$$\varphi_{A,B} : \Omega_A \rightarrow \Omega_B.$$

With the version of theorem 3.1 for λ taken into account, theorem 1.2 for λ reduces to

Theorem 5.1. Let $A = \{i_1, \dots, i_s\}$. For each $\omega \in \Omega_{A_\lambda}$ we have

$$\tau_\omega = \sum_{\omega' \in \varphi_{\emptyset, A_\lambda}^{-1}(\omega)} \tau_{\omega'}.$$

Proof. Consider the sets $A = \{i_1 < \dots < i_s\}$ and $B = A \cup \{i_{s+1}\}$ with $i_{s+1} > i_s$. We will prove the following formula by induction on s :

$$\tau_{\omega_B} = \sum_{\omega_A \in \varphi_{A,B}^{-1}(\omega_B)} \tau_{\omega_A}. \quad (7)$$

The stament of the theorem clearly follows.

Let λ_B be a weight with set of zero coordinates B and Π_B be the corresponding polytope. Furthermore, let $\omega_B \in \Pi_B$ and u_B be the corresponding vertex of Π_B . If $u_B \notin \Gamma_{i_{s+1}-1}$ (as facet of Π_B), then $\Gamma_{i_{s+1}-1} \not\subset \omega_B$ and $\omega_B \in \Omega_A$. Hence $\varphi_{A,B}^{-1}(\omega_B) = \{\omega_B\}$ and the formula is trivial.

If, however, $u_B \in \Gamma_{i_{s+1}-1}$, then note that $\{\Gamma_i, \Delta_i\} \subset \omega_B$ and apply theorem 2.3 to the cone $C_{u_B}^m$ and hyperplane $\Gamma_{i_{s+1}-1}$. More accurately, since the theorem considers only the finite dimensional case, we apply it to the cone $C_{u_B}^m$ with $m > l_{u_B}$ and the corresponding m -dimensional section of $\Gamma_{i_{s+1}-1}$. It is easily seen that the argument below shows that the obtained equality is actually (7) multiplied by

$$\prod_{i > m} (1 - \exp \gamma_i).$$

Clearly, if an edge of $C_{u_B}^m$ is not contained in $\Gamma_{i_{s+1}-1}$, then it is contained in one of Γ_i, Δ_i , hence the set of facets containing it supplemented by $\{\Gamma_{i_{s+1}-1}\}$ is of the form $\omega_A \in \varphi_{A,B}^{-1}(\omega_B)$ and the corresponding summand is equal to $\exp^*(u_B) \tau_{\omega_A}$. Conversely, any $\omega_A \in \varphi_{A,B}^{-1}(\omega_B)$ contains $\Gamma_{i_{s+1}-1}$ and corresponds to some edge of our cone.

It should be noted that this argument does not cover the case of $s = 0$ and $i_1 = 1$. However, this case is easily reduced to the case $s = 0$ and $i_1 = n$. Namely, $\Omega_{\{1\}}$ is obtained from $\Omega_{\{n\}}$ by substituting each ω for

$$\{\Gamma_1, \Delta_1\} \cup \{\Gamma_{i+1} | \Gamma_i \in \omega\} \cup \{\Delta_{i+1} | \Delta_i \in \omega\}.$$

□

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